MATCHING-PERFECT AND COVER-PERFECT GRAPHS

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ABSTRACT

It is shown that a graph G has all matchings of equal size if and only if for every matching set λ in G, $G \setminus V(\lambda)$ does not contain a maximal open path of odd length greater than one, which is not contained in a cycle. ($V(\lambda)$ denotes the set of vertices incident with some edge of λ.) Subsequently edge-coverings of graphs are discussed. A characterization is supplied for graphs all whose minimal covers have equal size.

1. Introduction and definitions

Let $G = (V, E)$ be a finite graph. A subset of E is a *matching set* if no two edges of the set are adjacent. A matching set is a *matching* in G if it is maximal in G with respect to inclusion. In a finite graph different matchings may have different cardinalities. In [1] Grünbaum uses $\underline{m}(G)$ and $\bar{m}(G)$ to denote the smallest and the greatest cardinalities respectively of matchings in G. In the above-mentioned paper Grünbaum displays all connected graphs G for which $\underline{m}(G) = \bar{m}(G) = 2$ and at the same time raises the problem of characterizing those graphs G for which $\underline{m}(G) = \bar{m}(G)$. Let us call such graphs *matching-perfect*, in brief MP.

For $\lambda \subset E$, let $V(\lambda)$ denote the set of vertices incident with some edge of λ. If $V(\lambda) = V$, then λ is called an *edge-cover* or, in short, a *cover* of G. A *minimum cover* of G is a cover of G having a minimum number of edges. It then follows directly from a result in [2] that a graph G is MP if and only if every matching of G is contained in a minimum cover of $G \setminus I(G)$, where $I(G)$ is the set of isolated vertices of G.

We now suggest a different characterization of MP graphs. We first introduce some definitions. A maximal open path in G of the form $(a_0, a_1, \cdots, a_n)$ such

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that $(a_0, a_n) \notin E$ is called a *snake* in G. The *length* of the snake is the number of its edges. A snake is *odd* or *even* according to the parity of its length. If λ is a matching set and $G \setminus V(\lambda)$ contains a snake s, then λ is a *blocking* of s in G. If s is a snake in G, then λ may be taken as $\emptyset$. s itself is referred to as a *blocked* snake. $|S|$ will denote the number of elements of the set S.

2. Matching-perfect graphs

THEOREM 1. *A graph is matching-perfect if and only if it does not contain an odd blocked snake.*

PROOF. We show that a graph is not MP if and only if it contains an odd blocked snake. The proof uses the familiar technique of "alternating paths".

Let $s = (a_0, a_1, \cdots, a_k)$ be an odd blocked snake of G, with λ a blocking of s in G. Put

$$\alpha = \{(a_0, a_1), (a_2, a_3), \cdots, (a_{k-1}, a_k)\}$$

and

$$\beta = \{(a_1, a_2), \ (a_3, a_4), \cdots, (a_{k-2}, a_{k-1})\}.$$

We see that $\alpha \cup \lambda$ is a matching set in G which may be completed by a set μ of edges to a matching $A = \alpha \cup \lambda \cup \mu$ of G. But then, since neither a_0 nor a_k are adjacent to any vertex in $G \setminus V(\alpha \cup \lambda)$, it follows that $B = \beta \cup \lambda \cup \mu$ is also a matching of G. But $|A| = |B| + 1$, so G is not MP.

Conversely, let G be such that $\underline{m}(G) < \bar{m}(G)$, and let A and B be matchings of G with $|A| > |B|$. Let C denote the family of maximal alternating chains in G, with edges alternatingly in A and in B. If all elements of C were to contain an even number of edges, we would have $|A| = |B|$. Hence there is an alternating path $s = (a_0, a_1, \cdots, a_k)$ of odd length k, that starts with an edge in A. If (a_0, x) is an edge of G with $x \notin V(\alpha)$, where α is defined as above, then $(a_0, x) \notin A \cup B$ since A is a matching and s is maximal, but $x \in V(B)$ since B is a matching; similarly for a_k. Let B^* denote the edges of B not in s; then it is clear that s is an odd blocked snake in G, with B^* a blocking of s in G.

The characterization obtained above, although not descriptive, suggests a method to detect a non-matching-perfect graph, which in many cases proves quite efficient.

3. Cover-perfect graphs

Let G be a graph without isolated vertices. Let $\underline{M}(G)$ and $\bar{M}(G)$ be the smallest and the greatest number of edges, respectively, in a minimal cover of

G. Let those graphs G for which $\underline{M}(G) = \bar{M}(G)$ be called *cover-perfect*, CP in short. We shall assume henceforth that G has no isolated vertices. A subset λ of E is *essential* in G if $G \setminus \lambda$ is without isolated vertices and $\bar{m}(G \setminus \lambda) < \bar{m}(G)$. We then have the following characterization of CP graphs.

THEOREM 2. *A graph G is cover-perfect if and only if it has no essential set of edges.*

This theorem follows quite easily from [2, Th. 2(i)].

It is quite easy to see that an MP graph has no essential edge. It does not follow, however, that an MP graph is necessarily CP. For example the complete graph K_n is MP for all n, but is CP only if $n < 4$. However, when G is a tree, we have:

THEOREM 3. *If a tree is matching-perfect, then it is cover-perfect.*

PROOF. Let $L(v)$ denote the set of edges incident with the vertex v. We first prove a lemma.

LEMMA. *If G is a tree and λ a subset of E such that for every vertex v of G we have $L(v) \not\subset \lambda$, then there is a matching set η in G such that $\eta \cap \lambda = \emptyset$ and $V(e) \cap V(\eta) \neq \emptyset$ for every $e \in \lambda$. Moreover there is an η for which $|\eta| = |\lambda|$.*

PROOF. Let x_0 be an end vertex of G. For each $(y, y') \in \lambda$, if y' disconnects G between x_0 and y, we choose an x such that $(x, y) \notin \lambda$, and include (x, y) in η. Clearly η thus constructed satisfies the requirements of the lemma.

Now suppose the tree G is MP and not CP. Then by Theorem 2, G has an essential set. Denote it by λ. λ satisfies the conditions of the lemma and hence there is a matching set η in G such that $\cap \lambda = \emptyset$ and $V(e) \cap V(\eta) \neq \emptyset$ for very $e \in \lambda$. Complete η to a matching η_0 in G. Since G is MP, we have $|\eta_0| = \bar{m}(G) \leqq \bar{m}(G \setminus \lambda)$. Then λ is not essential in G, contrary to our assumption. This proves the theorem.

COROLLARY. *Let G be a tree and λ an essential set in G. Then $|\lambda| < \frac{1}{2}|E|$.*

REFERENCES

1. B. Grünbaum, *Matchings in polytopal graphs*, Networks (to appear).
2. M. Lewin, *A note on line coverings of graphs*, Discrete Math. **5** (1973), 283–285.

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